following elegant result:

$$H(M, R) = H(M, M + 1 - R).$$

For instance, in Table 1 of the above letter[1] we have $H(6,2) = H(6,5) = 1.9729$.

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A Note on the Sampling of Zero Crossings of Two-Dimensional Signals

AVIDEH ZAKHOR AND DAVID IZRAELEVITZ

Curtis et al. applied a theorem due to Bezout to show that almost all continuous, periodic, band-limited two-dimensional signals can be reconstructed from at most $4(N_1 + N_2)^2$ zero-crossing samples where N_1 and N_2 is the number of Fourier coefficients in the signal. In this letter we prove a new version of Bezout's theorem and apply it to the above problem to provide a more lenient sampling requirement of at most $8N_1N_2$ zero-crossing samples.

INTRODUCTION

A considerable amount of research in the field of communication theory has been devoted to the problem of reconstruction of signals from their zero crossings [1]–[3]. Recently, Curtis *et al.* [1] applied Bezout's theorem to the problem of reconstructing a real, band-limited, continuous-time periodic two-dimensional signal $h(x, y)$ from a finite number of its zero crossings. Specifically if $F(n_1, n_2)$, the Fourier series coefficients of the signal, have a rectangular region of support given by

$$-N_1 \leqslant n_1 \leqslant N_1$$
$$-N_2 \leqslant n_2 \leqslant N_2$$

then almost all $h(x, y)$ can be uniquely determined from at most $4(N_1 + N_2)^2$ samples of its zero crossings. We will show that because of the geometry of the problem at most $8N_1N_2$ zero crossings are required and in the process we develop a tighter version of an important theorem in algebraic geometry, Bezout's theorem.

A TIGHTER VERSION OF BEZOUT'S THEOREM

Bezout's theorem is concerned with determining the number of common zeros of two bivariate polynomials and can be stated in the following manner:

Theorem 1 [5]: If two bivariate polynomials of degree r and s given by

$$f(x, y) = \sum_{i=0}^{r} \sum_{j=0}^{r-i} a(i, j) x^i y^j$$

$$g(x, y) = \sum_{i=0}^{s} \sum_{j=0}^{s-i} b(i, j) x^i y^j$$

have no common factor of degree greater than zero, then there exists at most rs common zeros.

A band-limited, continuous-time periodic signal $h(x, y)$ with periods T_1 and T_2 in x and y can be represented in terms of a polynomial $h'(w, z)$ in the variables $w = e^{j(2\pi x/T_1)}$ and $z = e^{j(2\pi y/T_2)}$

$$h(x, y) = w^{N_1} z^{N_2} h'(w, z)$$

$$h'(w, z) = \sum_{n_1=0}^{2N_1} \sum_{n_2=0}^{2N_2} F(n_1 - N_1, n_2 - N_2) w^{n_1} z^{n_2}. \quad (1)$$

Therefore using Bezout's theorem, if $h'(w, z)$ is irreducible, then $h'(w, z)$ (or equivalently $h(x, y)$) can be uniquely determined from $4(N_1 + N_2)^2$ zeros of $h'(w, z)$ (or $h(x, y)$). This is because the degree of $h'(w, z)$ in w and z is $2(N_1 + N_2)$ and Bezout's theorem deals with total degree rather than degree in each variable. In other words, it is concerned with polynomials whose coefficients have triangular support as shown in Fig. 1. On the other hand, one is usually interested in images with square or rectangular support in the Fourier domain where many of the coefficients corresponding to a triangular support are zero as shown in Fig. 2.

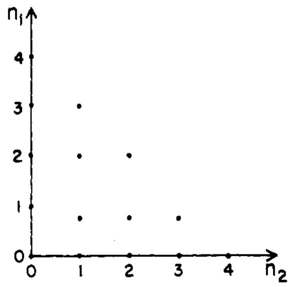

Fig. 1. Triangular support of a polynomial of degree 4.

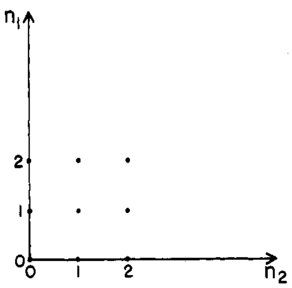

Fig. 2. Square support of a polynomial of maximum degree 2 in x and maximum degree 2 in y.

For the case when the polynomials under consideration have rectangular support, we are able to lower the bound on the number of common finite zeros from the bound set by Bezout's theorem. Specifically, if f and g are given by

$$f(x, y) = \sum_{i=0}^{N_x} \sum_{j=0}^{N_y} a(i, j) x^i y^j \quad (2)$$

$$g(x, y) = \sum_{i=0}^{M_x} \sum_{j=0}^{M_y} b(i, j) x^i y^j \quad (3)$$

the upper bound on the number of common finite zeros set by Bezout's theorem is $(N_x + N_y)(M_x + M_y)$. Our objective is to establish a tighter upper bound on the number of common finite zeros of f and g.

Before proceeding, we need to review several results concerning the *resultant* of polynomials in one or two variables. The resultant R_{pq} of two polynomials p and q in a single variable x

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N$$

Manuscript received February 3, 1986; revised March 21, 1986. This work was supported in part by the Advanced Research Projects Agency monitored by ONR under Contract N00014-81-K-0742 and in part by the National Science Foundation under Grant ECS-8407285. The work of A. Zakhor was supported by the Hertz Foundation.

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$$q(x) = b_0 + b_1x + b_2x^2 + \cdots + b_Mx^M$$

is defined [6] as the determinant of the $(M+N) \times (M+N)$ matrix

$$\begin{bmatrix} a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & a_n & 0 & 0 & \cdot & \cdot & 0 \\ 0 & a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & a_{n-1} & a_n & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & a_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_n \\ b_0 & b_1 & \cdot & \cdot & \cdot & b_m & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & b_0 & b_1 & \cdot & \cdot & b_{m-1} & b_m & 0 & \cdot & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & b_0 & \cdot & \cdot & \cdot & \cdot & \cdot & b_m \end{bmatrix}$$

A basic property of resultants is stated in the following theorem [6].

Theorem 2: When the polynomials p and q have numerical coefficients, a necessary and sufficient condition that they shall have a finite or infinite common root is that $R_{pq} = 0$.

Consider now two relatively prime bivariate polynomials $f(x, y)$ and $g(x, y)$ expressed as polynomials in x with coefficients which are polynomials in y

$$f(x,y) = a_0(y) + a_1(y)x + a_2(y)x^2 + \cdots + a_{N_x}(y)x^{N_x} \quad (4)$$

$$g(x,y) = b_0(y) + b_1(y)x + b_2(y)x^2 + \cdots + b_{M_x}(y)x^{M_x} \quad (5)$$

where each $a_i(y)$ is of degree at most N_y and each $b_i(y)$ is of degree at most M_y. We can define the resultant of f and g with respect to x as the determinant of the $(N_x + M_x) \times (N_x + M_x)$ matrix $M(y)$, with polynomial entries

$$\begin{bmatrix} a_0(y) & a_1(y) & \cdot & \cdot & \cdot & \cdot & \cdot & a_{N_x}(y) & 0 & 0 & \cdot & \cdot & 0 \\ 0 & a_0(y) & a_1(y) & \cdot & \cdot & \cdot & \cdot & a_{N_x-1}(y) & a_{N_x}(y) & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & a_0(y) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{N_x}(y) \\ b_0(y) & b_1(y) & \cdot & \cdot & \cdot & b_{M_x}(y) & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & b_0(y) & b_1(y) & \cdot & \cdot & b_{M_x-1}(y) & b_{M_x}(y) & 0 & \cdot & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & b_0(y) & \cdot & \cdot & \cdot & \cdot & \cdot & b_{M_x}(y) \end{bmatrix} \quad (6)$$

This resultant is a function of the remaining variable y and is denoted by $R_{fg}(y)$. Expanding the determinant of the above matrix, and taking into account that each $a_i(y)$ and $b_i(y)$ is of degree at most N_y and M_y, respectively, we can conclude that $R_{fg}(y)$ is a polynomial of degree $N_xM_y + M_xN_y$ or less. Moreover, it can be shown [5], that if $f(x, y)$ and $g(x, y)$ are relatively prime then $R_{fg}(y)$ is not identically zero. Thus the zero sets of f and g have at most $N_xM_y + M_xN_y$ values of y in common.

As our argument stands, we have not yet placed any tight limit on the number of intersections of f and g since for each y_i there could be a large number of x_j, such that for each j

$$f(x_j, y_i) = g(x_j, y_i) = 0. \quad (7)$$

In order to specify the number of x_j for each y_i, we need to study the behavior of $R_{fg}(y)$ in the vicinity of each y_i.

Theorem 3: If at each y_0 there are k values of x, x_j such that

$$f(x_j, y_0) = g(x_j, y_0), \qquad j = 1, \cdots, k$$

then the resultant of f and g with respect to x, $R_{fg}(y)$, has a zero of multiplicity k at y_0.

The above theorem implies that f and g as defined in (4) and (5) have at most $N_xM_y + M_xN_y$ intersections. Furthermore, $h(x, y)$ of (1) can now be uniquely determined using $8N_1N_2$ samples of its zero crossings instead of the usual $4(N_1 + N_2)^2$ as obtained via Bezout's theorem.

In order to prove Theorem 3 we need to review some results on matrix polynomial theory.

Theorem 4 (Smith Form [4]–[6]): Let $A(x)$ be an $n \times n$ polynomial matrix of rank r. We can find unimodular matrices $\{P(x), Q(x)\}$, such that

$$B(x) = P(x)A(x)Q(x)$$

and

1) *$B(x)$ is diagonal;*
2) *the first r diagonal elements of B* are monic polynomials $p_1(x), p_2(x), \cdots, p_r(x)$;
3) *the remaining diagonal elements, if any, are zero;*
4) *$p_i(x)$ divides $p_{i+1}(x)$ for $i = 1, 2, \cdots, r-1$.*

The *unimodular* polynomial matrices of the above theorem are defined to have nonzero constant determinant independent of x. Therefore, we get

$$|B(x)| = \prod_{i=1}^{r} p_i(x). \quad (8)$$

Also, from part 4) of Theorem 4 we can conclude that if $p_i(x) = 0$ then $p_k(x) = 0$ for $k \geq i$. From the above theorem, we can derive the following theorem:

Theorem 5: Let $A(x)$ be a polynomial matrix of full normal rank. If $A(x_0)$ has rank deficiency of k then, $|A(x)|$, the determinant of $A(x)$, has a zero of multiplicity k at x_0.

Normal rank in the above theorem is defined in [4]. A polynomial matrix $A(x)$ of full normal rank is one whose determinant is not identically zero for all values of x.

Proof: Using Theorem 4 we can find $B(x)$, the Smith normal form of $A(x)$. Since $P(x)$ and $Q(x)$ are unimodular, $B(x)$ is of full normal rank. Furthermore, the rank of $B(x)$ and $A(x)$ at each value of x, including x_0, is equal. Therefore $B(x_0)$ has rank deficiency of k and from (8) and part 4) of Theorem 4 $|B(x)|$ has a zero of multiplicity k at x_0. Since the determinant of $A(x)$ is within a constant factor of that of $B(x)$, $A(x)$ also has a zero of multiplicity k at x_0.

Using Theorem 5, we can now go on to show Theorem 3.

Proof of Theorem 3: Suppose that for y_0 there are k common finite zeros x_j, $j = 1, 2, \cdots, k$ between f and g. Then the matrix $M(y_0)$ defined by (6) must have k linearly independent null vectors given by

$$\begin{bmatrix} 1 & x_j & x_j^2 & \cdots & x_j^{N_x+M_x} \end{bmatrix}^T$$

for $j = 1, 2, \cdots, k$. Although $M(y)$ is of full normal rank (since f and g have no common factors), $M(y_0)$ has rank deficiency of k, and its determinant $R_{fg}(y_0)$ has a zero of multiplicity k at y_0. This completes the proof of Theorem 3.

From Theorem 3 we can derive the central result of this correspondence.

Theorem 6: If two bivariate polynomials of degree N_x and M_x in x and N_y and M_y in y given by (2) and (3) have no common factor of degree greater than zero, then there exists at most $N_xM_y + M_xN_y$ common finite zeros.

Proof: Since the resultant of f and g with respect to x is a polynomial of degree $N_xM_y + M_xN_y$, using Theorem 3 we can associate each finite common zero of f and g with a root of $R_{fg}(y)$. Since $R_{fg}(y)$ has at most $N_xM_y + M_xN_y$ roots, the theorem is proven.

Conclusion

We found an upper bound $N_x M_y + M_x N_y$ for the number of common finite zeros of two relatively prime polynomials given by (4) and (5). This bound is much tighter than $(N_x + N_y)(M_x + M_y)$ which is obtained via Bezout's theorem. For example, if the coefficients of a polynomial have region of support of $N \times N$, then it can be uniquely determined using $2N^2$ samples of its finite zero crossings as opposed to $4N^2$.

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On the Realization of Two's Complement Overflow Limit Cycle Free State-Space Digital Filters

VIMAL SINGH

A frequency-domain criterion for the overflow stability of a class of state-space digital filters is presented. The result is derived using a Lyapunov technique and the associated MKY machinery. Some consequences of the present derivation are discussed.

I. Introduction

Barnes and Fam [1] presented a "minimum norm" criterion for the absence of overflow oscillations in a class of state-space digital filters. Mills, Mullis, and Roberts [2] presented a generalized version of the criterion of [1]. Specifically, if there exists a positive diagonal matrix D such that $D - A^T DA$ is positive definite, then overflow oscillations are avoided. Kawamata and Higuchi [3] have carried out a Lyapunov analysis to derive a criterion which is analogous to the criterion of [2]. More recently, Bolton [4] has discussed realization, using Lyapunov method, of a two's complement overflow limit cycle free digital filter structure (of order two).

The purpose of this letter is to point out, employing a Lyapunov technique and the associated MKY machinery, a frequency-domain criterion for the overflow stability of the above-referred state-space digital filters. Some consequences of the present derivation are discussed in Section IV.

II. Frequency-Domain Criterion

The state-variable filter under consideration is of the form [1]–[4]

$$x(r+1) = f(y(r)) \tag{1a}$$

$$y(r) = Ax(r) \tag{1b}$$

where $x(r)$ is an n-vector state, A is the $n \times n$ coefficient matrix, and $f(y(r))$ is an n-vector nonlinear function, namely,

$$f(y(r)) = \begin{bmatrix} f_1(y_1(r)) \\ f_2(y_2(r)) \\ \vdots \\ f_n(y_n(r)) \end{bmatrix} \tag{2}$$

Manuscript received December 30, 1985; revised April 23, 1986.

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i.e., the n-vector $y(r)$ stands for the n-vector $Ax(r)$. It is assumed that

$$\det(zI - A) \neq 0, \qquad \text{for all } |z| \geq 1 \tag{3}$$

and that

$$\begin{aligned} &(A, I)\text{: completely controllable;} \\ &(A, A)\text{: completely observable} \end{aligned} \tag{4}$$

where I is the $n \times n$ identity matrix.

Assume a nonlinearity of the form

$$f_i(0) = 0, \qquad |f_i(y_i(r))| \leq |y_i(r)|, \quad i = 1, 2, \cdots, n. \tag{5}$$

The characteristic given in (5) includes, among others, an important arithmetic, namely, two's complement [1]–[4].

A criterion for the overflow stability of the above-described filter is given in the following theorem.

Theorem: (A sufficient condition for the absence of overflow oscillations.) For the zero solution of the circuit described by (1)–(5) to be asymptotically stable in the large, it is sufficient that there exists a positive diagonal matrix C such that the following is satisfied:

$$C + CA(zI - A)^{-1} + \left[CA(zI - A)^{-1}\right]^* \geq 0, \qquad \text{for all } |z| = 1 \tag{6}$$

where "$*$" denotes the conjugate transpose and "$\geq$" signifies that the matrix is positive semidefinite.

III. Derivation of the Stability Criterion

Consider a quadratic Lyapunov function

$$v(x(r)) = x^T(r) P x(r) \tag{7}$$

where $P = P^T$ is positive definite and T denotes the transpose. Application of (7) and (1a) results in

$$\begin{aligned} \Delta v(x(r)) &= v(x(r+1)) - v(x(r)) \\ &= f^T(y(r)) P f(y(r)) - x^T(r) P x(r). \end{aligned} \tag{8}$$

Adding to and subtracting from (8) the quantity

$$[f(y(r)) + y(r)]^T C [f(y(r)) - y(r)] \tag{9}$$

where C is a positive diagonal matrix, yields

$$\begin{aligned} \Delta v(x(r)) = &-x^T(r)[P - A^T CA]x(r) - f^T(y(r))[C - P]f(y(r)) \\ &+ [f(y(r)) + y(r)]^T C [f(y(r)) - y(r)] \end{aligned} \tag{10}$$

where (1b) has been utilized. With the substitution

$$f(y(r)) = y(r) - \tilde{f}(y(r)) \tag{11}$$

and employing (1b), (10) can be expressed as

$$\begin{aligned} \Delta v(x(r)) = &-x^T(r)[P - A^T PA]x(r) - x^T(r)[A^T P - A^T C]\tilde{f}(y(r)) \\ &- \tilde{f}^T(y(r))[PA - CA]x(r) \\ &- \tilde{f}^T(y(r))[C - P]\tilde{f}(y(r)) \\ &+ [f(y(r)) + y(r)]^T C [f(y(r)) - y(r)]. \end{aligned} \tag{12}$$

Set

$$P - A^T PA = QQ^T \tag{13a}$$

$$PA - CA = RQ^T \tag{13b}$$

$$C - P = RR^T \tag{13c}$$

where Q and R are real $n \times n$ matrices. Then (12) takes the form

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