SAMPLING SCHEMES FOR RECONTRUCTION OF MULTIDIMENSIONAL SIGNALS FROM MULTIPLE LEVEL THRESHOLD CROSSINGS

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ABSTRACT

It has been shown that under certain conditions multidimensional signals can be recovered from one-level crossings (e.g. zero crossings). However, the accuracy with which the locations of the one-level crossings need to be specified is large enough to limit its applicability in practical situations. To overcome this problem, we derive two sampling strategies for reconstruction of multidimensional signals from multiple level threshold crossings. We then propose two reconstruction algorithms for each of the two sampling schemes, and present a preliminary investigation of their quantization characteristics.

1 Introduction

Signal reconstruction in one and higher dimensions from zero crossings has been an active area of research [2,3,4]. Recently, Curtis *et.al.* showed that bandlimited periodic (BLP) 2-D signals are uniquely specified by their zero crossing to within a scale factor. Representing a 2-D signal with one-level crossings requires only one amplitude bit, but, in theory, an infinite number of position bits. In practice, the locations of the one-level crossings must be specified extremely accurately for successful reconstruction. On the other hand, representation of signals via their samples at the Nyquist rate requires few position bits and infinite (theoretically) amplitude bits. In this paper, we will develop intermediate sampling schemes which bridge the gap between Nyquist sampling and one-level crossing representation by enabling us to recover signals from multiple level threshold crossings. To this end, we derive semi-implicit and implicit[1]sampling strategies in section 2, and their corresponding reconstruction algorithms in section 3. Section 4 includes a preliminary investigation of the quantization characteristics of some of our proposed sampling and reconstruction schemes. More detailed discussions of the topics presented in this paper are included in [1].

2 Theoretical Results

Our approach is to represent BLP signals of the form

$$f(x,y) = \sum_{k_1=-N}^{N} \sum_{k_2=-N}^{N} F(k_1,k_2) e^{j2\pi(k_1 x + k_2 y)} \quad (1)$$

in terms of the polynomial

$$g(W_1,W_2) = f(x,y) W_1^N W_2^N \quad (2)$$

by letting $W_1 = e^{j2\pi x}, W_2 = e^{j2\pi y}$. Since reconstruction of $f(x,y)$ is equivalent to finding the coefficients of the polynomial $g(W_1,W_2)$, results from multivariate polynomial interpolation theory can be directly applied to a variety of multidimensional reconstruction problems. Unlike the univariate case, interpolation with multivariate polynomials is a non-trivial task. Whereas n arbitrary samples of a 1-D polynomial of degree $n-1$ are sufficient to find its coefficients, the analogous result in dimensions higher than one does not hold. In sections (2.1) and (2.2), we propose two schemes to circumvent this problem.

2.1 Semi-implicit Sampling Approach

Semi-implicit sampling strategies are based on multivariate interpolation results in which the locations of the interpolation points are restricted. Our main result in bivariate interpolation theory which is considerably less restrictive and more general than the earlier ones [5] can be stated in the following manner:

Theorem 1 *Consider the bivariate polynomial*

$$p(w,z) = \sum_{i=0}^{n_w} \sum_{j=0}^{n_z} a(i,j) w^i z^j$$

Let $c_0, c_1, ..., c_l$ be distinct bivariate irreducible polynomials with the maximum degrees of c_i in w and z given by $m_w^{(i)}$ and $m_z^{(i)}$, and l being an integer satisfying either $n_w < \sum_{i=0}^{l} m_w^{(i)}$ or $n_z < \sum_{i=0}^{l} m_z^{(i)}$. Define A_i to be the set of interpolation points on c_i given by

$$A_i = \{(w_j^{(i)}, z_j^{(i)}) \mid c_i(w_j^{(i)}, z_j^{(i)}) = 0, \ 0 \le j < S(i)\} \quad (3)$$

where $S(i) \equiv m_z^{(i)}(n_w - \sum_{k=0}^{i-1} m_w^{(k)}) + m_w^{(i)}(n_z - \sum_{k=0}^{i-1} m_z^{(k)}) + 1$. If none of the points given by (3) are on the intersections of two or more of the c_i's, then for any data set, we can uniquely interpolate $p(w,z)$.

The proof is based on a modified form of Bezout's theorem [6]. Theorem (1) requires irreducibility of the polynomials.

*This work has been supported in part by the Advanced Research Projects Agency monitored by ONR under Contract No. N00014-81-K-0742, in part by the National Science Foundation under Grant ECS-8407285, and in part by the John and Fannie Hertz Foundation.

[1] An implicit sampling scheme is one such as zero crossings for which the sampling coordinates are determined by the signal. A semi-implicit scheme is similar but, the sampling coordinates are additionally constrained by a prespecified function.

Two classes of polynomials which are known to be irreducible, and are particularly useful in deriving sampling strategies for multidimensional signals are of the form:

$$W_2^{M_y} = \alpha W_1^{M_x}, \quad W_2^{M_y} W_1^{M_x} = \alpha, \quad M_x, M_y > 0 \quad (4)$$

where M_y and M_x are positive integers which are relatively prime with respect to each other[2]. Using the relationship between W_1 and x, and between W_2 and y, and letting $\alpha = e^{j2\pi\beta}$, we conclude that the curves in the $W_1 - W_2$ plane given by equation (4) correspond to lines with rational slope of the form

$$M_y y = \beta + M_x x, \quad M_y y + M_x x = \beta, \quad M_x, M_y > 0 \quad (5)$$

in the $x - y$ plane. We can use this, together with equation (2) and Theorem (1) to define a semi-implicit sampling strategy for 2-D signals. More specifically, if the distribution of the samples of a 2-D BLP signal along lines of rational slope satisfies the conditions of Theorem (1), we can uniquely reconstruct it. An obvious way to apply this corollary to the problem of reconstruction from multiple level crossings is to choose the interpolation points at the intersection of level crossing contours and the sampling lines.

2.2 Implicit Approach

A primary drawback of the semi-implicit approach is that we can not guarantee that there will be enough intersections between sampling lines and the threshold contours to satisfy the conditions of theorem (1). This is particularly true, if the number of thresholds is small. To overcome this difficulty, we propose the implicit sampling approach, which is based on conditionally regular interpolation. Conditionally regular methods are uniquely solvable for most selections of interpolation points, but not all of them. Our main theoretical result can be stated in the following manner:

Theorem 2 *Consider a real, BLP, 2-D signal given by equation (1). Almost any $k > 0$ samples of its level crossings at α and $(2N+1)^2 - k$ samples of its level crossings at $\beta \neq \alpha$ are sufficient for its unique reconstruction provided that the following two conditions are satisfied:*
1. The sets $A_\alpha(R) \equiv \{(x,y) \in R^2 \mid f(x,y) = \alpha\}$ and A_β (defined in a similar fashion) are of maximal topological dimension.
2. The polynomials

$$g_{\alpha(\beta)}(W_1, W_2) = \sum_{k_1=0}^{2N} \sum_{k_2=0}^{2N} F_{\alpha(\beta)}(k_1 - N, k_2 - N) W_1^{k_1} W_2^{k_2}$$

$$F_{\alpha(\beta)}(k_1, k_2) = \begin{cases} F(0,0) - \alpha(\beta) & k_1 = k_2 = 0 \\ F(k_1, k_2) & elsewhere \end{cases}$$

are irreducible over the set of complex numbers.

The proof is included in [1]. The first condition of the above theorem requires the α and β level crossings of the signal to consist of at least one curve, and not isolated points. The second condition is also easily satisfied in practice. This is because the set of reducible polynomials with conjugate symmetric coefficients have been shown to be of measure zero in the set of polynomials with conjugate symmetric coefficients.

[2]As shown in [1], for the special case when the irreducible interpolation curves are of the form $W_2 = \alpha W_1^m$, there is an alternate proof to Theorem (1), resulting in a recursive algorithm for determining the coefficients of the polynomial.

Theorem (2) can be easily extended to recovery of multidimensional signals from more than two threshold crossings. More specifically, for m distinct thresholds, $t_1, ..., t_m$, almost any distribution of $(2N+1)^2$ points among the thresholds will result in unique reconstruction of the signal under consideration, provided that the level crossings have maximal topological dimensions, and their associated polynomials are irreducible.

3 Reconstruction Algorithms

A complete survey of the reconstruction algorithms for the semi-implicit and implicit sampling strategies, and their corresponding examples are included in [1]. In this paper, however, we will briefly describe two classes of reconstruction algorithms.

The most straightforward approach to reconstruction from semi-implicit or implicit samples of level crossings is to solve a linear system of equations (possibly overdetermined) in order to determine the Fourier series coefficients of the signal under consideration. Most linear least-squares algorithms are extremely storage and computation intensive. This is because reconstructing a signal with $(2N+1) \times (2N+1)$ region of support in the Fourier domain requires storage of a $(2N+1)^2 \times (2N+1)^2$ matrix.

To circumvent the storage and computational problems of the linear least-squares approach, we have developed iterative algorithms for reconstruction of signals from their level crossings. The iterative algorithm imposes space and frequency domain constraints on the signal in an iterative fashion. The steps of the iterative algorithm for the implicit sampling strategy can be described in the following manner:

1. Assume that all the crossing contours of the p thresholds $t_1 < ... < t_p$ associated with a signal of the form given by equation (1) are quantized in position on an $M \times M$ grid where $M > (2N+1)$. If the intensity of the signal lies in the range $[t_0, t_{p+1}]$, then the quantized threshold contours can be used to derive the intensity range of the signal on the nodes of a $M \times M$ grid:

$$t_i(n_1, n_2) \le f(\frac{n_1}{M}, \frac{n_2}{M}) \le t_{i+1}(n_1, n_2), \quad 0 \le n_1, n_2 < M$$

2. Let $f^{(l)}$ denote the solution in the lth iteration, and choose an arbitrary initial guess $f^{(0)}$.
3. Take the DFT of $f^{(l)}$ to get $F^{(l)}$.
4. Impose the bandlimited constraint:

$$\tilde{F}^{(l+1)}(k_1, k_2) = \begin{cases} (1 - \lambda_1) F^{(l)}(k_1, k_2) & , N < k_1, k_2 < M - N \\ F^{(l)}(k_1, k_2) & , elsewhere \end{cases}$$

5. Take the inverse DFT of $\tilde{F}^{(l+1)}$ to get $\tilde{f}^{(l+1)}$.
6. Impose the space domain constraint

$$f^{(l+1)} \equiv \begin{cases} \tilde{f}^{(l+1)} & , \ t_i \le \tilde{f}^{(l+1)} \le t_{i+1} \\ \tilde{f}^{(l+1)}[1 - \lambda_2] + \lambda_2 t_i & , \ \tilde{f}^{(l+1)} < t_i \\ \tilde{f}^{(l+1)}[1 - \lambda_2] + \lambda_2 t_{i+1} & , \ \tilde{f}^{(l+1)} > t_{i+1} \end{cases}$$

7. If all the nodes of the $M \times M$ grid satisfy the space domain constraint, we are done. Otherwise repeat steps (3) through (6).

We can use the theory of projection onto convex sets (POCS) [7] to show that the algorithm converges to a solution satisfying both the space and frequency domain constraints, provided $0 < \lambda_1$ *and* $\lambda_2 < 2$. The simplest way to accelerate the convergence is over-relaxation, which involves setting $1 < \lambda_i < 2$.

The iterative algorithm for the semi-implicit sampling strategy reconstructs the 1-D signals associated with each sampling line in a iterative fashion, and then interpolates the 2-D signal from 1-D ones. The basic idea behind this algorithm is the fact that the 1-D signal obtained by sampling a 2-D BLP signal along a line of rational slope $\frac{n}{m}$ is itself BLP. Thus, if all the intersections of level crossings and the sampling lines are known, we can deduce the intensity range for equally spaced points on the line. This space domain constraint together with the frequency domain constraint resulting from bandlimitedness of the 1-D signals can be used to derive an iterative scheme for reconstruction of the 1-D signals. In a manner similar to the iterative algorithm for the implicit sampling case, we can apply the theory of POCS to establish the convergence of this algorithm, and determine ways of accelerating it.

After the 1-D signals associated with sampling lines are determined, they can be used to interpolate the 2-D signal under consideration. If the value of a signal of the form given by equation (1) is needed on a $(2N+1) \times (2N+1)$ grid, one strategy is to first determine the value of the signal at the intersections of sampling lines and $(2N+1)$ equally spaced horizontal or vertical lines, then reconstruct the 1-D signals associated with these vertical or horizontal ones, and finally, determine the signal at $(2N+1)$ equally spaced points on each horizontal or vertical lines.

4 Preliminary Investigation of Quantization Properties

A rigorous and thorough investigation of the quantization properties of our various sampling and reconstruction schemes involves extensive consideration of coding issues and experiments, and has not yet been carried out. However, based on some preliminary experiments, some tentative conclusions and speculations are possible which can be used as a starting point for further research in the areas of multidimensional signal representation and image coding.

To limit the computational intensity of the preliminary investigation, we have chosen to carry out our experiments on only one picture with 15×15 region of support in the Fourier domain. We begin with quantization characteristics of the linear least-squares method.

4.1 Linear Least-Squares Method

As shown in [1], for the linear least-squares approach, the number of reconstruction samples affects the reconstruction robustness to a great extent. Specifically, increasing the number of samples beyond the minimum number required by the theoretical results initially decreases the required number of position bits per sample. Beyond a certain point however, it increases the total number of position bits required to represent a signal. Thus, we have chosen to carry out our experimental investigation at a fixed number of reconstruction samples. In addition, while our quantization strategy for implicit sampling has been to quantize the coordinates of the samples, we have chosen to specify locations of semi-implicit samples by quantizing their positions along their sampling lines, and specifying the lines that they fall on.

Figures (1a) and (1b) show the plot of mean square error (mse) versus the normalized number of amplitude and position bits as a function of the number of thresholds, for reconstruction via the linear least-squares method and QR decomposition. The mse between the original and reconstructed image is given by:

$$mse = \frac{1}{(2N+1)^2} \sum_{k_1=-N}^{N} \sum_{k_2=-N}^{N} [F(k_1,k_2) - \tilde{F}(k_1,k_2)]^2 \quad (6)$$

where $F(k_1,k_2)$ and $\tilde{F}(k_1,k_2)$ correspond to the Fourier coefficients of the original and reconstructed image respectively. We have found experimentally that the quality of the reconstructed image becomes almost indistinguishable from the original one when $mse < .1$. Normalized number of position and amplitude bits is defined to be the total number of amplitude and position bits (for all the reconstruction samples) normalized to the number of Fourier coefficients. By definition, the number of amplitude bits is given by $\log_2(1 + \textit{number of thresholds})$. Figure (1b) corresponds to the implicit sampling strategy and (1a) corresponds to semi-implicit sampling strategy with equidistant lines of unit slope. In both cases, the number of samples used for reconstruction was 256. The slope of the curves shown in figures (1a) and (1b) are negative, indicating that the quality of the reconstruction is improved as the the number of position bits is increased. In addition, the spacing between the curves decreases from right to left, indicating that the improvement in the quality of reconstruction decreases as the the number of thresholds becomes larger. The most interesting feature of the semi-implicit curves of figure (1a) is that increasing the number of thresholds from 7 to 16 improves the quantization characteristics, while further increase from 32 to 64 thresholds, degrades it. Similarly for the implicit sampling curves of figure (1b), the "optimum" number of thresholds which results in minimum number of bits varies as a function of mse. For instance, for $.2 \leq mse \leq 1$ it is 7, for $.06 \leq mse \leq .2$ it is 16, and for $.02 \leq mse \leq .06$ it is 32.

4.2 Iterative Methods

We begin with characterization of the semi-implicit approach.

4.2.1 Semi-implicit Sampling

Figure (1c) shows the mean square error between our test image and its reconstructed version via the iterative algorithm , versus the number of position and amplitude bits. The number of thresholds associated with the four curves is 6, 8, 12, and 16. For each curve the thresholds were chosen with equal spacing between 0 and 256. The four points on each curve correspond to different numbers of equally spaced points on the sampling lines, i.e. $M = 32, 64, 128$. In addition, the sampling lines were chosen to be equidistant and of unit slope. The y axis corresponds to mse, as defined by equation (6), and the x axis indicates normalized number of position and amplitude bits used.

Because of the inherent structure of the samples used by the iterative algorithm, there are a variety of ways to represent the signal under consideration and to arrive at the

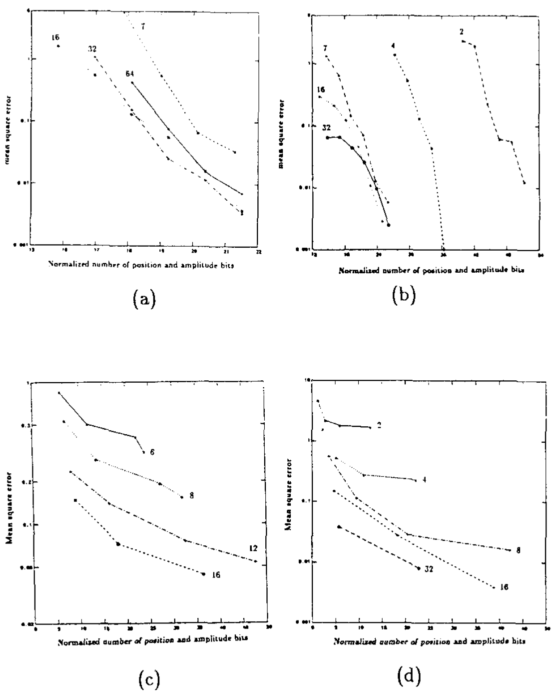

Figure 1: *Plot of mse versus normalized number of position and amplitude bits as a function of the number of thresholds for the linear least-squares method with (a) 256 semi-implicit samples; (b) 256 implicit samples, and the iterative algorithm with: (c) semi-implicit samples; (d) implicit samples.*

number of quantization bits. The most straightforward way is to quantize the location of threshold crossings on the sampling lines in a similar fashion to quantization in the linear least-squares method, and derive the space domain constraint from the quantized samples. Alternatively, we can specify the intensity range for each of the equally spaced points on each sampling line. Our strategy in determining the total number of position bits for the abscissa of figure (1c) has been to choose the minimum of the above two quantization strategies.

As seen in figure (1c), the slope of each curve is negative indicating that for a fixed number of thresholds the mean square error decreases as M is increased. In addition, for fixed M, the mean square error is decreased as the number of thresholds is increased. The decreasing distance between the curves shown in figure (1c) is indicative of the fact that as the number of thresholds increases, the resulting drop in mse decreases. Thus, the decrease in mse as the number of thresholds changes from 6 to 8 is more substantial than when it changes from 12 to 16. An interesting question to address, however, is whether or not there is an "optimum" number of thresholds for which the lowest number of amplitude and position quantization bits is achieved. As figure (1c) shows, this "optimum" number varies as a function of the mean square error. For instance, for the value of mse in the range $[0.53, 0.85]$, it is between 6 to 8, and for mse in the range $[0.15, 0.23]$, it is between 12 and 16.

4.2.2 Implicit Sampling

Similar to the iterative reconstruction algorithm of the semi-implicit sampling, there are several strategies one might take to arrive at the total number of amplitude and position quantization bits for representing a given image via the implicit sampling scheme. One way to encode the boundary points of a quantized threshold contour is to do contour tracing. Alternatively, we can specify the range of the signal for each node of the $2^b \times 2^b$ quantization grid. Our adopted quantization strategy, which is almost certainly not optimal, has been to choose the minimum of these two quantization strategies for representing images.

Figure (1d) shows the plot the mean square error versus *normalized number of position and amplitude bits as a function* of the number of thresholds. The five curves of figure (1d) correspond to reconstruction from different number of thresholds. Various points on each curve correspond to reconstruction with different values of grid size. The slopes of the curves are negative, indicating that the quality of reconstruction improves as the quantization grid becomes finer. In addition, the number of thresholds which results in smallest number of quantization bits is a function of mse. For instance, if we are interested in reconstructing signals with $mse \leq .556$, then the optimal number of thresholds is between 8 and 16. Finally, comparing figures (1c) and (1d), it seems that for fixed quality of reconstruction via iterative algorithms, implicit sampling results in lower number of bits than semi-implicit sampling with lines of unit slope.

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