Two-Dimensional Polynomial Interpolation From Nonuniform Samples

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***Abstract*—We derive a number of results on sufficient conditions under which the two-dimensional (2-D) polynomial interpolation problem has a unique or nonunique solution. We find that unless an appropriate number of interpolation points are chosen on an appropriate number of irreducible curves, the resulting problem might become singular. Specifically, if the sum of the degrees of the irreducible curves on which the interpolation points are chosen is small compared to the degree of the interpolating polynomial, then the problem becomes singular. Similarly, if there are too many points on any of the irreducible curves on which the interpolation points are chosen, then the interpolation problem runs into singularity. Examples of geometric distribution of interpolation points satisfying these conditions are shown. The examples include polynomial interpolation of polar samples, and samples on straight lines. We propose a recursive algorithm for computing 2-D polynomial coefficients for the nonsingular case where all the interpolation points are chosen on lines passing through the origin. Finally, we apply our result to the problem of nonuniform frequency sampling design for 2-D FIR filter design, and show a few examples of such design.**

I. Introduction

Nonuniform sampling is of importance in many signal processing problems such as filter design, speech processing, power spectral estimation, holography, astronomy, and data compression [1]. In most of these problems, either uniform samples are not available due to practical reasons, or variations in the instantaneous bandwidth of a signal necessitates nonuniform sampling rates corresponding to local characteristics of the signal.

Nonuniform samples in the frequency domain have been used for the design of finite impulse response (FIR) filters [2]–[5]. The basic idea is that the transfer function of a one-dimension (1-D) filter is a 1-D polynomial of a finite order and therefore can be reconstructed from a finite number of nonuniform frequency samples.

While 1-D polynomial interpolation techniques have been extensively applied to filter design and other signal processing problems [6], [7], there has been little work in the area of multidimensional (M-D) interpolation. This can be attributed to the fact that a number of mathematical results in 1-D do not hold in two or higher dimensions. For instance, the fundamental theorem of algebra which guarantees factorizability of polynomials in 1-D, does not hold in two or more dimensions. In fact, it has been shown that most 2-D polynomials are irreducible [8].

Another such example is polynomial interpolation. Unlike the univariate (one-dimensional) case, interpolation with multivariate polynomials is a nontrivial task. Whereas N arbitrary samples of a univariate polynomial of degree $N - 1$ are sufficient to find its coefficients, the analogous result in dimensions higher than one does not hold, primarily because Chebyshev systems in R^s for $s \geq 2$ have been shown not to exist. Chebyshev systems are important in interpolation theory, and have been extensively studied by many researchers including Karlin [9], Karlin and Sudden [10], and Krein [11]. A linearly dependent set of continuous functions $\{u_0(x), \cdots, u_N(x)\}$ defined on $[a, b]$ is a Chebyshev system if for any $a \leq x_0 < \cdots < x_N \leq b$ and $y_0, \cdots, y_N \in R$, there is a unique linear combination $u(x) = \Sigma_{j=0}^{N} a_j u_j(x)$ satisfying $u(x_i) = y_i$ for $i = 0, \cdots, N$. Clearly, the set of N continuous functions consisting of the powers of x form a Chebyshev system. Another example of a Chebyshev system is $u_i(x) = e^{\lambda_i x}$, where λ_i are distinct and $x \in (-\infty, +\infty)$ [12].

Although Chebyshev systems are helpful in studying univariate interpolation, we must leave them behind as soon as we turn to multivariate interpolation. This is because there are no sets of N universal functions which can be used for interpolation at any N distinct points [13]. An implication of this result is that powers of x or y do not form a Chebyshev system in R^2, and thus, bivariate polynomials are not in general uniquely reconstructible from their samples at arbitrary locations. Although it can be argued that almost all random selections of interpolation points result in a unique solution [4], there are two major problems with this approach. First, from a numerical analysis point of view, the condition number of the resulting inverse problem might become too large, thus resulting in unstable interpolation. Second, in many applications, the interpolation points are likely to be chosen on well-defined geometric objects such as lines or circles, and therefore are not randomly distributed. As we will see, this can greatly enhance the likelihood of running into singularities.

In this paper, we derive conditions under which the 2-D interpolation problem is guaranteed to have a unique or

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nonunique solution, develop a recursive algorithm for a class of interpolation points, and apply our results to a 2-D FIR filter design problem.

II. Results on Unique Interpolation

Our approach in deriving conditions for unique 2-D interpolation is to constraint the locations of interpolation points on irreducible curves.[1] Specifically, we have

Theorem 1: A bivariate polynomial of the form

$$p(w, z) = \sum_{i=0}^{N_w} \sum_{j=0}^{N_z} a(i, j) w^i z^j \tag{1}$$

is uniquely reconstructible from samples on $n_c + 1$ *distinct, irreducible curves provided there are a minimum of*

$$S(i) \equiv M_z^{(i)}\left(N_w - \sum_{k=0}^{i-1} M_w^{(k)}\right) + M_w^{(i)}\left(N_z - \sum_{k=0}^{i-1} M_z^{(k)}\right) + 1$$

samples on the ith curve, with no samples on intersection of any two curves. The maximum degrees of the ith curve in w and z are defined to be $M_w^{(i)}$ *and* $M_z^{(i)}$*, respectively. The number of required sampling curves,* $n_c + 1$ *is defined to be an integer satisfying either of the following two inequalities:*

$$N_w < \sum_{i=0}^{n_c} M_w^{(i)} \qquad N_z < \sum_{i=0}^{n_c} M_z^{(i)}. \tag{2}$$

The proof is included in Appendix A. An example of the distribution of sampling points required by this result for $N_w = N_z = 2, M_w^{(0)} = M_z^{(0,1)} = 1$ and $M_w^{(1)} = 2$ is shown in Fig. 1. The irreducible curves in this example were chosen to be of the forms $wz = \alpha_0$ and $z^2 = \alpha_1 w$. In general, determining irreducibility of polynomials is a nontrivial task [15]. However, two classes of polynomials which are known to be irreducible, and are particularly useful in deriving our results, are of the form

$$z^{M_z} = \alpha w^{M_w} \qquad z^{M_z} w^{M_w} = \alpha \tag{3}$$

where M_z and M_w are relatively prime positive integers. As we will see in Section IV, we can apply the above result in conjunction with Theorem 1 to nonuniform frequency sampling design of 2-D FIR filters.

While Theorem 1 provides conditions for unique interpolation,[2] it does not specify an algorithm for the actual interpolation. The counterpart of such an algorithm in 1-D uses divided differences or Newton's method [16]. The most straightforward, but not so elegant, way of determining the polynomial coefficients in 2-D is to solve a linear (possibly overdetermined) system of equations, which by Theorem 1 is guaranteed to have a unique solution. The inherent disadvantage of solving a linear least squares (LLS) problem is its computational complexity. Specifically, robust linear least square algorithms such as the ones based on QR decomposition require $O(N^6)$ operations for a 2-D polynomial with $N \times N$ coefficients [17]. On the other hand, if we impose a specific structure on the location of the frequency samples, we can devise less computationally intensive algorithms. An example of this is the case in which there is one sampling curve of the form $z = \alpha w^N$ with $(N + 1)^2$ sampling points. Under these conditions the problem of finding filter coefficients reduces to that of solving $(2N + 1)^2 \times (2N + 1)^2$ Vandermonde equations which can be solved with as little as $O(N^4)$.

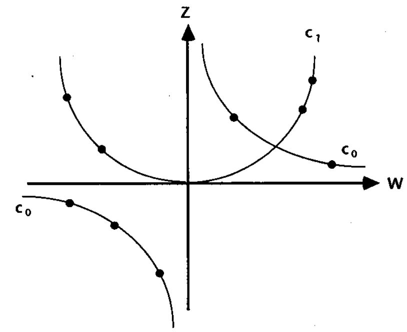

Fig. 1. An example of geometric distribution of the interpolation points for Theorem 1 with $N_w = N_z = 2$ and $M_w^{(0)} = M_z^{(0,1)} = 1$, and $M_w^{(1)} = 2$. The irreducible curve c_0 is of the form $wz = \alpha_0$, and c_1 is of the form $z^2 = \alpha_1 w$.

Similarly, for the special case where all the irreducible curves are chosen to be lines passing through the origin, we can derive a recursive algorithm for computing the coefficients. The basic idea behind this algorithm is that substitution of equations of these lines in the equation of the 2-D polynomial results in a series of 1-D polynomials. Specifically, consider an $N \times N$ polynomial with $N + 1$ interpolation lines $l_0, l_1, \cdots, l_N$ of the form $z = \alpha_i w$ with $\alpha_i \neq 0$ and $2i + 1$ points on the ith line. Defining the variables $b_i^{(k)}$ associated with the kth sampling line in terms of its slopes α_k and the polynomial coefficients $a(i, j)$ in the following way:

$$b_i^{(k)} = \begin{cases} \sum_{m=0}^{i} a(i-m, m)\alpha_k^m & 0 \le i \le N \\ \sum_{m=i-N}^{N} a(i-m, m)\alpha_k^m & N \le i \le 2N \end{cases} \tag{4}$$

we conclude that samples on the kth line satisfy

$$\sum_{i=s+1}^{2N-s-1} b_i^{(k)} w^i = p(w, \alpha_k w) - \sum_{i=2N-s}^{2N} b_i^{(k)} w^i. \tag{5}$$

The recursive algorithm essentially consists of $2N + 1$ steps corresponding to $s = -1, \cdots, 2N - 1$ in the

[1] By irreducible curves, we mean curves whose algebraic equations are not factorable.

[2] Even though Theorem 1 deals with unique recovery of polynomial coefficients, we will use the term "unique interpolation" rather than "unique reconstruction" throughout this paper. This has to do with the fact that in the mathematics community where most multidimensional interpolation results appeared originally, unique interpolation is the terminology used for unique specification or reconstruction of the polynomial coefficients.

above equation. Specifically, at the sth stage, the algorithm uses the computed values of $b_{2N-s}^{(k)}$ and $b_s^{(k)}$ for $0 \leq k \leq N$, together with the $2(N - s) - 1$ samples on the line l_{N-s-1}, in order to find $b_i^{(N-s-1)}$ for $s + 1 \leq i \leq 2N - s - 1$ by solving a Vandermonde linear system of equations indicated by (5). These values are then used in (4) to find the coefficients $a(i, j)$ for $i + j = s + 1$ and $i + j = 2N - s - 1$ by solving a second set of linear Vandermonde equations. Finally, these newly determined coefficients are used to find $b_{2N-s-1}^{(k)}$ and $b_{s+1}^{(k)}$ for $0 \leq k \leq N - s - 2$ via (4) for the next step of the algorithm. Sizes of the first and second set of Vandermonde equations in the sth step are $(2N - s) \times (2N - s)$ and $(s + 2) \times (s + 2)$, respectively. Since stable techniques for solving $N \times N$ Vandermonde equations require $O(N^3)$ operations, the number of flops for recovering the coefficients of an $(2N + 1) \times (2N + 1)$ filter via the proposed recursive method is of the order of $O(N^4)$.

The extension of the above algorithm to the case where all the sampling lines have identical integer slope is included in Appendix B.

III. Results on Nonunique Interpolation

In the previous section, we discussed sufficient conditions for unique recovery of polynomials from their samples on irreducible curves. We will now find conditions under which the interpolation problem is guaranteed to run into singularity.

Theorem 2: Let $c_0, c_1, \cdots, c_p$ be distinct, irreducible bivariate polynomials with the maximum degree of c_i in w and z given by $M_w^{(i)}$ and $M_z^{(i)}$ where p is an integer satisfying the following two inequalities:

$$N_w > \sum_{i=0}^{p} M_w^{(i)} \qquad N_z > \sum_{i=0}^{p} M_z^{(i)}. \tag{6}$$

If the interpolation points are chosen on $c_0, \cdots, c_p$, then the polynomial coefficients of (1) cannot be uniquely determined.

The proof is included in Appendix C. Intuitively speaking, the above theorem implies that if the sum of the degrees of the irreducible curves on which the interpolation points are chosen is small compared to the degree of the polynomial, then the interpolation problem becomes singular and therefore has a nonunique solution. Three examples of geometric distribution of the points corresponding to the above theorem are shown in Fig. 2. In Fig. 2(a), $N_w = N_z = 2$ and the algebraic equation for the irreducible curve c_0 is given by $wz = \alpha_0$. Note that Fig. 2(a) can be obtained from Fig. 1 by moving all the points on c_1 to c_0, and that in doing so, we convert a nonsingular interpolation problem to a singular one. A second example of Theorem 2 is shown in Fig. 2(b) where the degree of the interpolating polynomial is $N_w = N_z = 3$, and the sampling curves are chosen to be lines. As seen, unless the number of sampling lines is large compared to the degree of interpolating polynomial, the problem of interpolation from samples on lines could become singular. The third example of Theorem 2 is shown in Fig. 2(c) where the degree of interpolating polynomial is assumed to be 5 in w and in z and the interpolation curves are chosen to be circles. From Fig. 2(c) it is clear that in order to avoid running into singularities, the degree of 2-D polynomials interpolating polar samples must be chosen appropriately with respect to the number of circles.

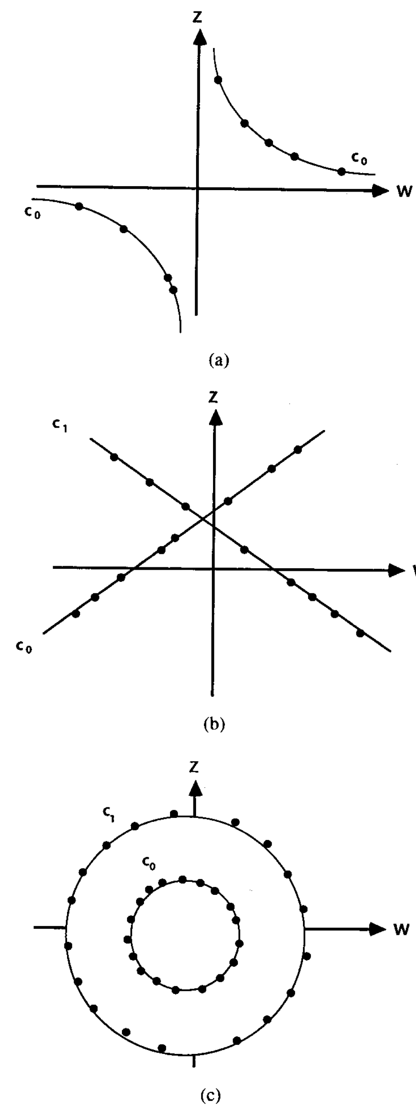

Fig. 2. Three examples of geometric distribution of the interpolation points for Theorem 2.

Our next result is also on conditions under which the 2-D interpolation problem runs into singularity:

Theorem 3: Consider a polynomial of the form shown in (1) with $(N_w + 1)(N_z + 1)$ interpolation points. If there is an irreducible curve, c_0, of the form

$$z^{M_z} = \alpha w^{M_w} \quad \text{or} \quad z^{M_z} w^{M_w} = \alpha \tag{7}$$

which contains more than $M_z N_w + M_w N_z + 1$ interpolation points, then the coefficients of the polynomial cannot be uniquely determined.

The proof is included in Appendix D. The above theorem states that if there are too many points on any one of the irreducible curves on which the interpolation points are chosen, then the interpolation problem runs into singularity. Intuitively, if the equation of the curve shown in (7) is substituted into the 2-D polynomial equation (1), a 1-polynomial results. In effect, too many points on this curve corresponds to overspecification of its corresponding 1-D polynomial, and hence nonunique interpolation. Two examples of geometric distribution of the interpolation points corresponding to the above theorem are shown in Fig. 3. In Fig. 3(a), the degree of the interpolating polynomial is chosen to be $N_w = N_z = 2$, c_0 is of the form $wz = \alpha_0$, and c_1 is of the form $z = \alpha_1 w^2$. Note that the distribution of the points in Figs. 1 and 3(a) is very similar in a sense that by moving one point from c_1 to c_0, we obtain the latter from the former. Nonetheless, one set corresponds to nonsingular and the other corresponds to a singular interpolation problem. The second example corresponding to Theorem 3 is shown in Fig. 3(b) where the degree of the interpolating polynomial is chosen to be $N_w = N_z = 3$ and the irreducible sampling curves consist of a line and a circle.

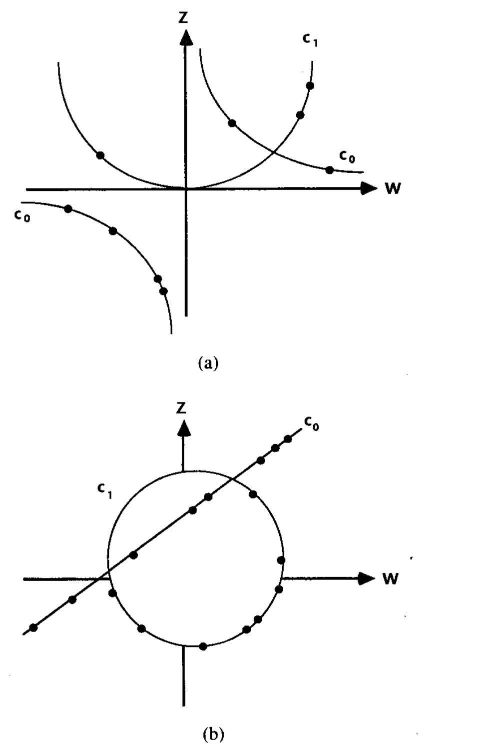

Fig. 3. Two examples of geometric distribution of the interpolation points for Theorem 3.

IV. Application to 2-D FIR Filter Design

Let the frequency response of a 2-D FIR filter with impulse response $h(n_x, n_y)$ be given by

$$H(e^{j\omega_x}, e^{j\omega_y}) = \sum_{n_x=-N_x}^{N_x-1} \sum_{n_y=-N_y}^{N_y} h(n_x, n_y) \exp j(\omega_x n_x + \omega_y n_y) \tag{8}$$

and the polynomial representation of the above frequency response be given by

$$H(w, z) = w^{N_x} z^{N_y} H(e^{j\omega_x}, e^{j\omega_y}) = \sum_{n_x=0}^{2N_x} \sum_{n_y=0}^{2N_y} h(n_x - N_x, n_y - N_y) w^{n_x} z^{n_y} \tag{9}$$

where $w = e^{-j\omega_x}$ and $z = e^{-j\omega_y}$. The relationship between (w, z) and (ω_x, ω_y) implies that irreducible polynomials of the form shown in (3) correspond to lines with positive or negative rational slopes of the form $M_y \omega_y = \beta + M_x \omega_x$ in the $\omega_x - \omega_y$ plane. We can use this in conjunction with Theorem 1 in order to define a nonuniform frequency sampling technique in which the samples are chosen on lines of rational slope in the $\omega_x - \omega_y$ plane. Specifically, we have the following.

Corollary 1: Coefficients of a 2-D FIR filter of the form given in (8) can be uniquely determined from frequency samples on $N_l + 1$ lines in the $\omega_x - \omega_y$ plane provided there are a minimum of $S(i) = |M_y^{(i)}|(2N_x - \Sigma_{k=0}^{i-1} |M_x^{(k)}|) + |M_x^{(i)}| 2(N_y - \Sigma_{k=0}^{i-1} |M_y^{(k)}|) + 1$ samples on the ith line. The slope of the ith line is rational and is given by $M_x^{(i)}/M_y^{(i)}$. The number of required sampling curves, $N_l + 1$ is defined to be an integer satisfying either of the following inequalities:

$$2N_x < \sum_{i=0}^{N_l} |M_x^{(i)}| \qquad 2N_y < \sum_{i=0}^{N_l} |M_y^{(i)}|. \tag{10}$$

Furthermore, if the total number of interpolation points $\Sigma_{i=0}^{N_l} S(i)$ is equal to the number of filter coefficients, then the frequency response of the interpolated filter passes through prescribed values at the sampling points.

The above corollary provides an exact description of the distribution of the frequency sampling points required for unique specification of the filter coefficents. Specifically, it states that for a given filter size, if an appropiate number of sampling points on an appropriate number of lines with rational slope are chosen, then the filter coefficients can be uniquely determined. It also specifies conditions under which the designed filter actually takes on prescribed values at prescribed points. This could be important in applications in which specific spatial frequencies need to be removed or emphasized. Design parameters of the nonuniform frequency sampling technique based on Corollary 1 include: a) the number of the sampling lines and their slopes; b) the distribution of the sampling lines in the frequency plane; and c) the distribution of the frequency samples on each line. A similar nonuniform frequency sampling technique has already been applied to the 2-D FIR filter design problem [4], [5]. This technique, however, is a special case of Corollary 1 since the locations of the frequency samples in [4] are constrained to be on parallel horizontal or vertical lines.

An example of the geometric distribution of the sampling points required by the corollary for $N_x = N_y = 1$ $M_x^{(0)} = -1$, $M_y^{(0)} = 1$, $M_x^{(1)} = 1$, $M_y^{(1)} = 2$ is shown in Fig. 4. Note that the curves of Fig. 1 in the $w - z$ domain correspond to lines of Fig. 4 in the $\omega_x - \omega_y$ domain. In addition, since the frequency response is a periodic function in the $\omega_x - \omega_y$ plane, lines of rational slope "wrap around" or extend modulo 2π in ω_x and ω_y throughout the frequency domain. This is illustrated in Fig. 4 where lines l_0 and l_1 with slopes -1 and 2 are wrapped around once and twice, respectively.

Plots of frequency responses corresponding to a 15 × 15 circular low-pass filter with passband at 0.4π and stopband at 0.6π designed via the recursive and LLS approaches are shown in Figs. 5(a) and 6(a). The distribution of the sampling lines and sampling points for these two cases are shown in Figs. 5(b) and 6(b). The sampling points are more or less uniformly distributed along each line, except that for lines passing through the stopband and passband, there is a minimum of three equally spaced points in the transition band. The values of the transition points for the interpolation problem are chosen linearly from 0 to 1. There are sampling points at the intersection of passband/stopband contours and the sampling lines. The total number of sampling lines for both the recursive and LLS schemes is 15. However, the sampling lines for the LLS approach have slopes ± 1, while those of the recursive approach have slope 1. This is because the recursive approach can only be applied to lines of identical slope in the $\omega_x - \omega_y$ domain or equivalently to lines passing through origin in the $w - z$ plane. As seen in Fig. 6(c), since the sampling lines for the LLS case run in two different directions, we can place frequency sampling points all over the perimeter of the passband and stopband contours. This feature does not exist in Fig. 5(c) because all the lines run along the same direction, and therefore the sampling points can only exist along half the perimeter of stopband/passband contours. This difference manifests itself in a slightly better isocontour shapes and passband/stopband maximum deviation values for the LLS approach than the recursive approach, even though our technique is not designed to optimize these values. The isocontours for the LLS and recursive approach for $|H(e^{j\omega_x}, e^{j\omega_y})| = 0.3, 0.5, 0.7, 0.9$ are shown in Figs. 6(d) and 5(d), respectively. As seen, the isocontours of the recursive approach are less circular in the lower left and upper right of the frequency plane, whereas those of the LLS approach are more or less isotropic.

A thorough performance comparison between our technique and uniform sampling requires extensive testing for a variety of filter specifications, and as such is an exhaustive task. Nevertheless, we have obtained some preliminary results on performance comparison of the filters in

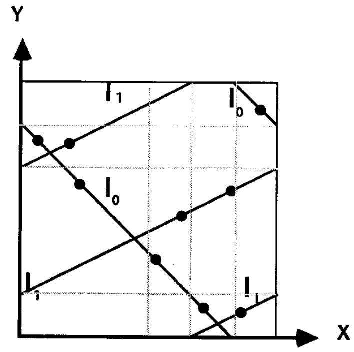

Fig. 4. An example of geometric distribution of the frequency samples of Corollary 1. The sampling curves in $(w - z)$ domain in Fig. 1 correspond to lines in $(\omega_x - \omega_y)$ plane in Fig. 4.

Fig. 6(a) and a 15 × 15 uniform frequency sampling filter. The frequency response and isocontours of the latter filter at 0.3, 0.5, 0.7, and 0.9 are shown in Figs. 7(a) and (b). Comparing Figs. 7(b) and 6(d) we conclude that the nonuniform sampling results in more circular contours than uniform sampling, particularly at the 0.9 level. Quantitatively, this can be explained by noting that the stopband and passband ripples for uniform sampling are 0.036 and 0.060, while those of the nonuniform samples are 0.043 and 0.036.

Similar to Theorem 1, Theorems 2 and 3 can be used to derive conditions under which the nonuniform frequency sampling technique does not result in unique filter coefficients. Specifically, Theorem 2 implies that if the number of sampling lines of unit slope is fewer than 15, then the coefficients of the 2-D 15 × 15 filter cannot be uniquely determined. Theorem 2 also implies that if the sampling lines are chosen to be of slope 2 (or $\frac{1}{2}$), then the minimum number of required sampling lines to avoid singularity is 8. Theorem 3, on the other hand, implies that if the total number of interpolation points is 225, then no one line of slope $+1$ or -1 could have more than 15 frequency samples.

An interesting point to notice is that our proposed nonuniform frequency sampling technique can be combined with the transformation technique [18]–[20]. Specifically, it can be used to design a low-order filter which is then frequency transformed via a one-dimensional optimal filter to result in a high order two-dimensional FIR filter. An example of this for a directional filter with ideal frequency response of the form [21]

$$H(e^{j\omega_x}, e^{j\omega_y}) = \begin{cases} 1 & \dfrac{\pi}{8} < \tan^{-1}\left(\dfrac{\omega_x}{\omega_y}\right) < \dfrac{\pi}{4} \text{ and} \\ & 0.4\pi < (\omega_x^2 + \omega_y^2) < 0.9\pi \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

is shown in Fig. 8(a). The transition bandwidth used in designing the filter is 0.2π. The entire "hybrid" two-di-

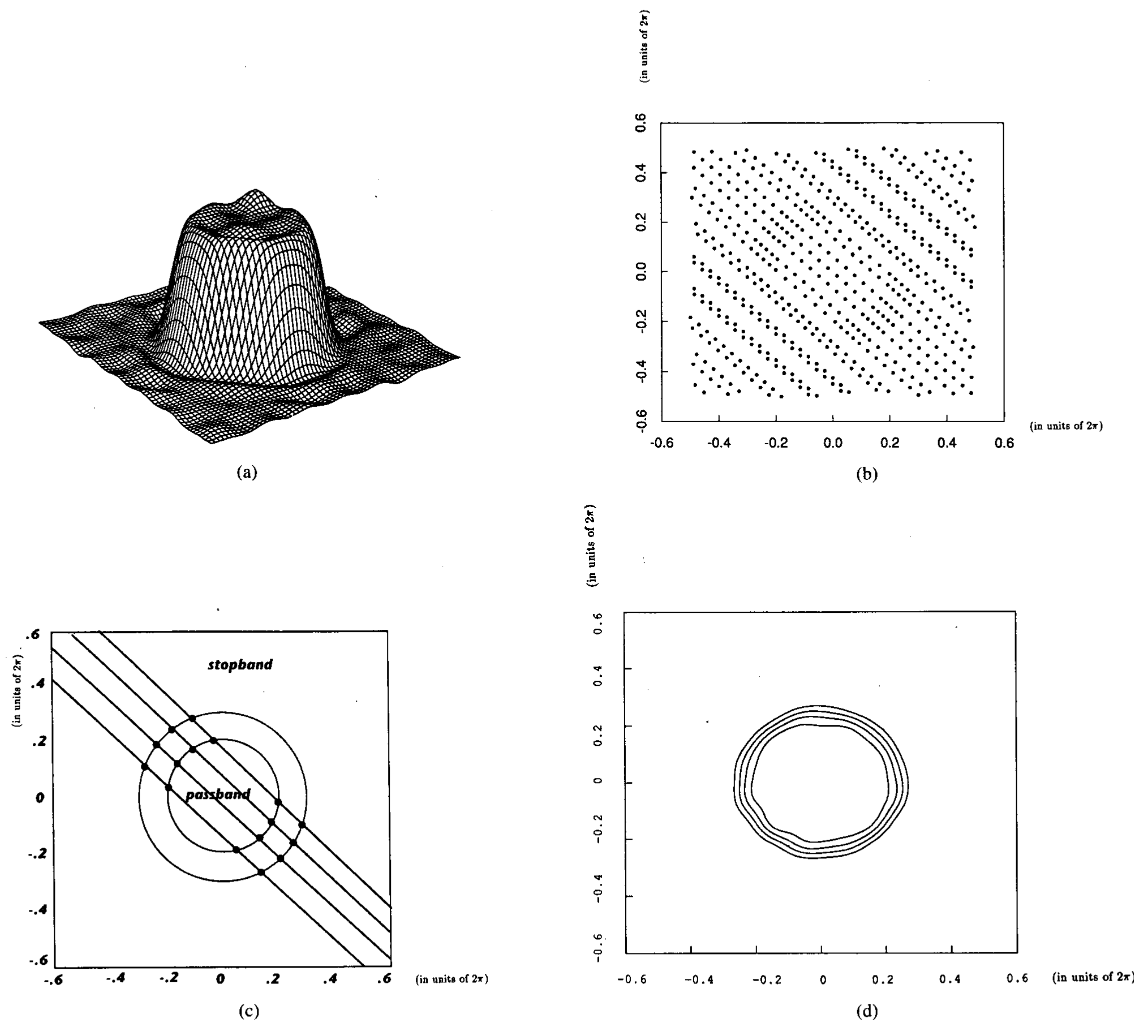

Fig. 5. (a) The frequency response of a 15 × 15 low-pass filter designed via Corollary 1 and the recursive algorithm. (b) The distribution of the frequency samples used for interpolation. (c) Intersection of sampling lines and passband/stopband contours. (d) Isocontours of the filter in (a).

mensional filter is 19 × 19, the low-order two-dimensional filter designed via our proposed technique is 3 × 3, and the optimal, minimum ripple, one-dimensional filter is of length 7. The low-order 2-D filter was designed by sampling the ideal frequency response shown in (11) along three lines of unit slope.

Our last example is a 15 × 15 directional filter designed via the nonuniform frequency sampling technique of Corollary 1 with the LLS approach, shown in Fig. 8(b). As seen in Fig. 8(c), 13 of the sampling lines are chosen to be of slope -1 and two are chosen to have slope $+1$. The sampling points are more or less uniformly distributed along each line, except that for lines passing through the stopband and passband, there is a minimum of three equally spaced points in the transition band. Furthermore, as seen in Fig. 8(d), there is a sampling point at the intersection of passband/stopband contours and the sampling lines. The isocontours of the filters in Figs. 8(a) and (b) are shown in Figs. 8(e) and (f). As seen, even though the passband and stopband deviation characteristics of the transformation filter is superior to that of the nonuniform frequency sampling, the isocontours of the latter are more similar to the shape of the specifications for passband and stopband contours.

V. Discussion

We derived a number of results on conditions under which the 2-D polynomial interpolation problem has unique or nonunique solution. Our approach consisted of finding appropriate constraints on the locations of the interpolation points. We proposed a recursive algorithm for computing 2-D polynomial coefficients for the case where all the interpolation points are chosen on lines passing through origin. Finally, we applied our result to the prob-

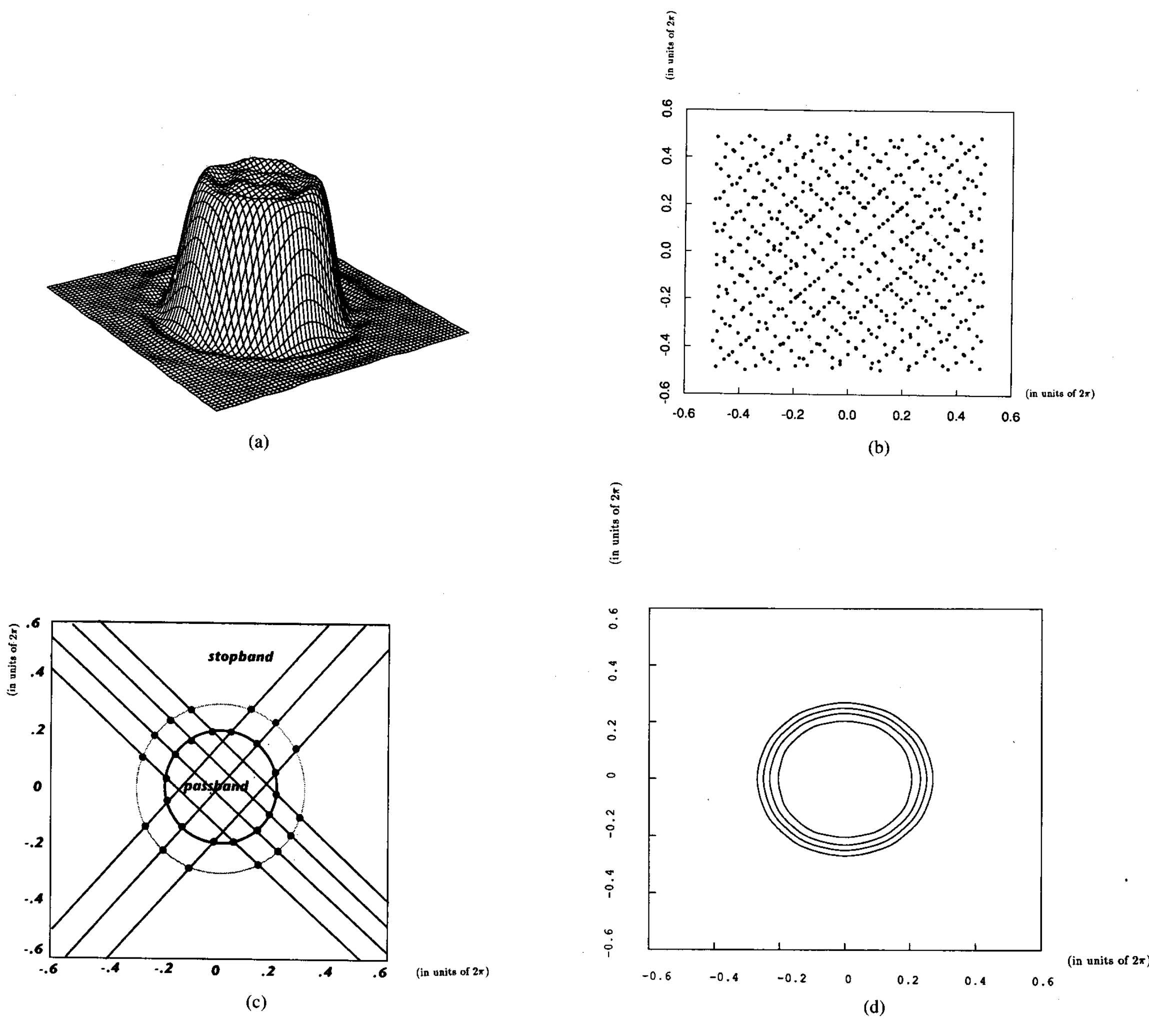

Fig. 6. (a) The frequency response of a 15 × 15 low-pass filter designed via Corollary 1 and the LLS approach. (b) The distribution of the frequency samples used for interpolation. (c) Intersections of sampling lines and passband/stopband contours. (d) Isocontours of the filter in (a).

lem of nonuniform frequency sampling design for 2-D FIR filter design, and showed an example of such a design.

We found that unless an appropriate number of interpolation points are chosen on an appropriate number of irreducible curves the 2-D polynomial interpolation problem might run into singularities. Specifically, Theorem 2 implies that if the sum of degrees of the irreducible curves on which the interpolation points are chosen is small compared to the degree of the polynomial, then the interpolation problem becomes singular. Theorem 3 states that if there are too many points on any of the irreducible curves on which the interpolation points are chosen, then the interpolation problem runs into singularity. Examples of geometric distribution of interpolation points satisfying these theorems were shown. Two important applications of these examples are polynomial interpolation of polar grid samples and samples on straight lines.

In applying the results to the 2-D FIR filter design problem, we found that an appropriate selection of frequency samples on an appropriate number of lines with rational slope results in a unique determination of the coefficients. Examples of low-pass and directional filters using the recursive and LLS approach were shown. To shape the isocontours in a desired fashion, the sampling points were chosen at the intersection of sampling lines and passband/stopband contours. Unlike the recursive approach which requires the sampling lines in the frequency plane to have identical slopes, the lines in LLS can be of any slope. As a result, we get slightly better contour shaping properties with the latter method.

The results presented here can be easily extended to dimensions larger than 2. Furthermore, they can be extended to the case where the interpolation points are on reducible curves, since by definition, a reducible curve

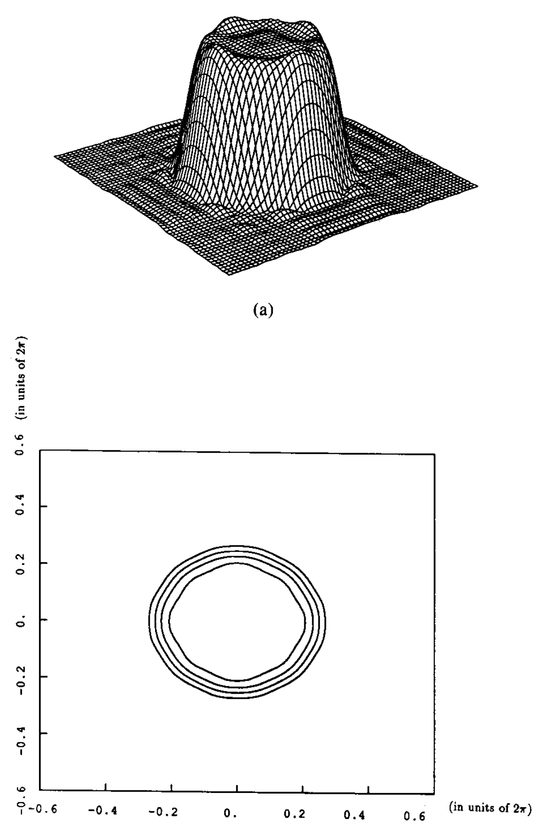

Fig. 7. (a) The frequency response of a 15 × 15 low-pass filter designed via uniform frequency sampling and inverse discrete Fourier transform. (b) Isocontours of the filter in (a).

can be factored into a number of irreducible ones. It is important to emphasize that the multidimensional polynomial interpolation results presented in this paper are important to many other areas of signal processing and systems problems as well as FIR filter design [22]. For instance, they have already been applied to nonuniform sampling of band-limited periodic signals [13], [23]. Possible directions for future research include modeling of multidimensional signals and systems.

Appendix A
Proof of Theorem 1

Proof: To show that there is a unique polynomial, we have to show that there are no polynomials in $\Pi_{(N_w, N_z)}$ which vanish at all the interpolation points $\cup A_i$. Suppose, on the contrary, that there is a polynomial $q(w, z) \in \Pi_{(N_w, N_z)}$ which vanishes at all the interpolation points. Since q has $M_w^{(0)} N_z + N_w M_z^{(0)} + 1$ common zeros with C_0, by the modified version of Bezout's theorem, C_0 must be a factor of $q(w, z)$. That is

$$q(w, z) = C_0(w, z) q^{(1)}(w, z)$$

where $q^{(1)}(w, z)$ is a polynomial of maximum degree $N_w - M_w^{(0)}$ in w and $N_z - M_z^{(0)}$ in z. Furthermore, since by hypothesis, none of the interpolation points on C_1 are on C_0 and $q(w, z)$ has $1 + M_z^{(1)}(N_w - M_w^{(0)}) + M_w^{(1)}(N_z - M_z^{(0)})$ common zeros with C_1, $q^{(1)}(w, z)$ must also have the same number of common zeros with C_1. Taking into account the irreducibility of C_1, by modified version of Bezout's theorem, C_1 must be a factor of $q^{(1)}(w, z)$ and hence $q(w, z)$.

Repeating the above argument for $C_2, \cdots, C_{N_c - 1}$, we get

$$q(w, z) = C_0(w, z) \cdots C_{N_c - 1}(w, z) q^{(N_c)}(w, z)$$

where $q^{(p)}(w, z)$ has maximum degree $N_w - \Sigma_{i=0}^{N_c - 1} M_w^{(i)}$ in w and maximum degree $N_z - \Sigma_{i=0}^{N_c - 1} M_z^{(i)}$ in z and has $1 + M_w^{(N_c)}(N_z - \Sigma_{k=0}^{N_c - 1} M_z^{(k)}) + M_z^{(N_c)}(N_w - \Sigma_{k=0}^{N_c - 1} M_w^{(k)})$ common zeros with C_{N_c}. Since C_{N_c} is irreducible, by modified version of Bezout's theorem, it must be a factor of $q^{(N_c)}(w, z)$. This contradicts the hypothesis since by inequalities (2), the degree of C_{N_c} in either w or z, is larger than that of $q^{(N_c)}(w, z)$. □

Appendix B
Generalized Recursive Algorithm

In this Appendix, we describe a recursive algorithm for finding coefficients of a polynomial of the form

$$p(w, z) = \sum_{i=0}^{N} \sum_{j=0}^{N} a(i, j) w^i z^j$$

from its samples on irreducible curves $c_0, c_1, \cdots, c_p$. The ith curve c_i is assumed to be of the form

$$z = \alpha_i w^m \qquad \alpha_i \neq 0$$

where $m < N$, and p is chosen to be the smallest integer such that

$$\sum_{i=0}^{p} [(m + 1)n - 2mi] \geq (N + 1)^2.$$

The set of interpolation points on c_i is given by

$$\{(w_j^{(i)}, z_j^{(i)}) | j = 0, \cdots, (m + 1)N - 2mi\}$$

and the interpolation points are assumed not to be on the intersection of any two curves.

The recursive algorithm consists of $p + 1$ steps. We will use induction to show that in the ith step we can find the $2mi$ coefficients given by the set

$$\{a(l_1, l_2) | l_1 + ml_2 = (i - 1)m, \cdots, im - 1, N(m + 1) - im + 1, \cdots, N(m + 1)\}.$$

In doing so, we exploit the fact that sampling the bivariate polynomial $p(w, z)$ along the ith curve, c_i is equivalent to sampling the univariate polynomial

$$p_i(w) = p(w, \alpha_i w^m) = \sum_{r=0}^{N(m+1)} b_r^{(i)} w^r \qquad (12)$$

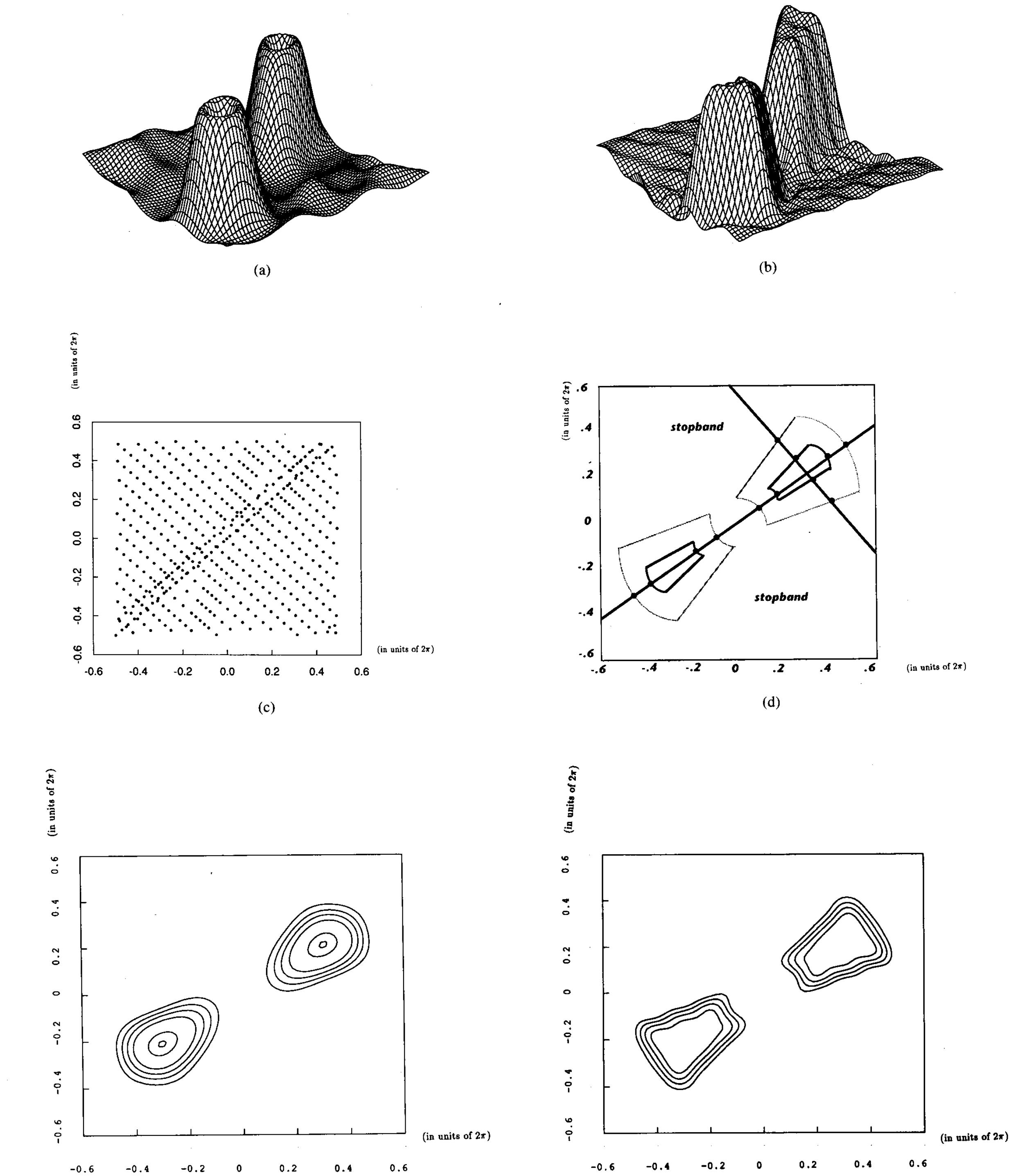

Fig. 8. (a) Frequency response of a 19 × 19 directional filter designed via the frequency sampling technique and McClellan's transformation. (b) Frequency response of a 15 × 15 directional filter designed via the sampling points in (c) using LLS approach. (c) Sampling points for the filter shown in (b). (d) Intersections of sampling lines and passband/stopband contours. (e) Isocontours of the filter in (a). (f) Isocontours of the filter in (b).

with

$$b_r^{(i)} = \sum_{\substack{l_1=0 \\ l_1+ml_2=r}}^{N} \sum_{l_2=0}^{N} \alpha_i^{l_2} a(l_1, l_2). \tag{13}$$

In the first step of the algorithm, we can use the points on the curve c_0 given by the set

$$\{(w_j, \alpha_0 w_j^{(0)}) \mid j = 0, \cdots, (m+1)N\}$$

in order to uniquely determine $b_i^{(0)}$ for $i = 0, \cdots, N(m+1)$. This is because $p_0(w)$ of (12) is a one-dimensional polynomial degree $N(m+1)$, and thus any $(m+1)N + 1$ distinct samples of it are sufficient for unique determination of its coefficients. We can now use the value of the quantities

$$\{b_i^{(0)} \mid i = 0, \cdots, m-1, N(m+1) - (m-1), \cdots, N(m+1)\}$$

together with (13) to find the coefficients

$$\{a(l_1, l_2) \mid l_1 + ml_2 = 0, \cdots, m-1, N(m+1) - (m-1), \cdots, N(m+1)\}.$$

The values of the remaining coefficients of the polynomial $p_0(w)$, which are found in the first step of the algorithm will be used in future steps. More specifically, for $j = 0, \cdots, m-1$ the quantities $b_{i+mj}^{(0)}$ and $b_{N(m+1)-im-j}^{(0)}$ will be used in the $(i+1)$st step of the algorithm.

Having shown the validity of the induction hypothesis for the first step of the algorithm, we will now show that if in steps 1 through i the quantities

$$\{a(l_1, l_2) \mid l_1 + ml_2 = 0, \cdots, im-1, N(m+1) - im + 1, \cdots, N(m+1)\}$$

and

$$\{b_r^{(j)} \mid j = 0, \cdots, i; \quad r = jm, jm+1, \cdots, N(m+1) - jm + 1\}$$

are found, then in the $(i+1)$st step the quantities

$$\{a(l_1, l_2) \mid l_1 + ml_2 = mi, \cdots, m(i+1) - 1, N(m+1) - (i+1)m, \cdots, N(m+1) - im\}$$

and

$$\{b_r^{(i+1)} \mid r = im, \cdots, N(m+1) - im\}$$

can be determined. Rearranging the terms in (12) we get

$$\begin{aligned} \tilde{p}(w, \alpha_i w^m) &= p(w, \alpha_i w^m) - \sum_{\substack{l_1=0 \\ l_1+ml_2=0, \cdots, m-1, N(m+1)-im+1, \cdots, N(m+1)}}^{N} \sum_{l_2=0}^{N} \cdot \alpha_i^{l_2} a(l_1, l_2) w^{l_1+ml_2} \\ &= \sum_{\substack{l_1=0 \\ l_1+ml_2=im, \cdots, N(m+1)-im}}^{N} \sum_{l_2=0}^{N} \cdot \alpha_i^{l_2} a(l_1, l_2) w^{l_1+ml_2} \\ &= \sum_{r=im}^{N(m+1)-im} b_r^{(i)} w^r. \end{aligned} \tag{14}$$

By hypothesis, since the point (0, 0) is on the intersection of all the curves $c_0, \cdots, c_p$, it could not possibly be one of the interpolation points. Therefore we have

$$w_j^{(i)} \neq 0 \qquad j = 0, \cdots, N(m+1) - 2im.$$

This implies that the points on the ith curve c_i given by the set

$$\{(w_j^{(i)}, \alpha_i w_j^{(i)}) \mid j = 0, \cdots, N(m+1) - 2mi\}$$

are sufficient to uniquely specify the coefficients

$$\{b_r^{(i)} \mid r = im, \cdots, N(m+1) - im\}$$

of the univariate polynomial given by (14). The values of b's found in the $(i+1)$st step of the algorithm together with the ones found in previous steps can now be used for finding the coefficients of $p(w, z)$. More specifically, for $j = 0, \cdots, m-1$ the quantities

$$\{b_{im+j}^{(k)} \mid k = 0, \cdots, i\}$$

can be used to find coefficients

$$\{a(l_1, l_2) \mid l_1 + ml_2 = mi + j\}.$$

Using (13) we have

$$b_{im+j}^{(k)} = \sum_{r=0}^{i} \alpha_k^r \, a(j + im - rm, r). \tag{15}$$

Since the curves $c_0, \cdots, c_p$ are distinct, the α's corresponding to different curves are different from each other. Therefore the coefficients

$$\{a(l_1, l_2) \mid l_1 + ml_2 = mi + j\}$$

can be uniquely found by solving the Vandermonde system of (15). The same procedure can be applied for finding

$$\{a(l_1, l_2) \mid l_1 + ml_2 = N(m+1) - im - j\}$$

from the quantities

$$\{b_{N(m+1)-im-j}^{(k)} \mid k = 0, \cdots, i\}.$$

Therefore, we have shown that in step $i+1$, we can uniquely determine the quantities

$$\{a(l_1, l_2) \mid l_1 + ml_2 = mi, \cdots, m(i+1) - 1, N(m+1) - (i+1)m + 1, \cdots, N(m+1) - im\}.$$

This completes the induction and the description of the algorithm.

Appendix C
Proof of Theorem 2

Proof: To show that there are an infinite number of polynomials, it is sufficient to show that there is at least one polynomial of maximum degree smaller than N_w in w and smaller than N_z in z which vanishes at all the intersection points defined by the theorem. The most obvious choice for this polynomial is $q(w, z) = \Pi_{i=0}^{p} c_i(w, z)$. Since by inequalities in (6), the maximum degree of $q(w, z)$ is less than N_w in w and less than N_z in z, there exists an infinite number of polynomials of degree N_w in w and N_z in z passing through the interpolation points. □

Appendix D
Proof of Theorem 3

Proof: To show that there are infinite number of polynomials, it is sufficient to show that the interpolation matrix associated with the interpolation points has dependent rows. The jth row of the interpolation matrix corresponds to the jth interpolation points and is given by

$$[1, w, w^2, \cdots w^{N_w}, z, zw, zw^2, \cdots, zw^{N_w}, \cdots, z^{N_z}w^{N_w}].$$

Our strategy is to show that the rows associated with the points on c_0 are dependent. To show this, consider the one-dimensional function $h(z)$[3] defined by inserting the algebraic equation associated with c_0 into $p(w, z)$ of (1). Since c_0 is of the form given by (7), $h(z)$ is in fact a polynomial in z^{1/M_w} or in $z^{-(1/M_w)}$ depending upon whether c_0 is of the form $z^{M_z} = \alpha w^{M_w}$ or $z^{M_z}w^{M_w} = \alpha$. Without loss of generality, assume $h(z)$ is a polynomial in z^{1/M_w}. If we let $h(z) = f(z^{1/M_w})$, then the polynomial $f(z)$ will be of degree $N_zM_w + N_wM_z$ in z, and hence can be uniquely specified from $N_zM_w + N_wM_z + 1$ samples. For instance, if c_0 is of the form $z_z^M = \alpha w^{M_y}$, then inserting $w = (\alpha^{-1}z^{M_z})^{1/M_w}$ into $p(w, z)$ of (1) we get

$$h(z) = \sum_{i=0}^{N_w}\sum_{j=0}^{N_z} a(i, j)\alpha^{-iM_w}z^{j+i(M_z/M_w)} \tag{16}$$

$$f(z) = h(z^{M_w}) = \sum_{i=0}^{N_w}\sum_{j=0}^{N_z} a(i, j)\alpha^{-iM_w}z^{j+iM_z}. \tag{17}$$

If Q denotes the number of interpolation points on c_0, the dimension of the interpolation matrix, $F_{Q\times(M_zN_w+M_wM_z+1)}$, associated with $f(z)$ is $Q \times (N_zM_w + N_wM_z + 1)$. Since by hypothesis $Q > N_zM_w + N_wM_z + 1$, the rank of F cannot exceed $N_zM_w + N_wM_z + 1$, hence making its rows linearly dependent. The interpolation matrix associated with $p(w, z)$ on the other hand is $(N_w + 1)(N_z + 1) \times (N_w + 1)(N_z + 1)$ dimensional, with each row corresponding to one interpolation point. Suppose the Q rows of this matrix corresponding to the Q points on c_0 form a $Q \times (N_w + 1)(N_z + 1)$ dimensional matrix which we denote by G. Careful examination of F and G shows that each column in F is a linear combination of a number of columns in G. This is because (16) and (17) imply that coefficients of $f(z)$ are linear combination of coefficients of $p(w, z)$. Therefore, linear dependency among the rows of F imply linear dependency among rows of G. This implies that the $((N_w + 1)(N_z + 1)) \times ((N_w + 1)(N_z + 1))$ dimensional interpolation matrix associated with $p(w, z)$ is rank deficient. Therefore, there are infinitely many polynomials interpolating the points specified by the theorem. □

[3]The same argument can be repeated for the variable w.

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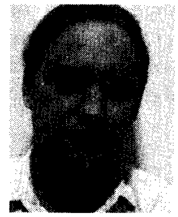

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